

Anderson's orthogonality catastrophe

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ABSTRACT. We give an upper bound on the modulus of the ground-state overlap of two non-interacting fermionic quantum systems with N particles in a large but finite volume L^d of d -dimensional Euclidean space. The underlying one-particle Hamiltonians of the two systems are standard Schrödinger operators that differ by a non-negative compactly supported scalar potential. In the thermodynamic limit, the bound exhibits an asymptotic power-law decay in the system size L , showing that the ground-state overlap vanishes for macroscopic systems. The decay exponent can be interpreted in terms of the total scattering cross section averaged over all incident directions. The result confirms and generalises P. W. Anderson's informal computation [Phys. Rev. Lett. **18**, 1049–1051 (1967)].

1. Introduction

Anderson's orthogonality catastrophe (AOC) is an intrinsic effect in many-body fermionic systems. It arises when a system reacts to a sudden perturbation. For instance, one may think of a sudden X-ray excitation of a core electron in an atom, leaving behind a hole in a core shell. In bulk metals the AOC manifests itself in the asymptotic vanishing

$$\langle \Phi_L^N, \Psi_L^N \rangle \sim L^{-\gamma/2} \quad (1.1)$$

of the overlap of the N -body ground states Φ_L^N and Ψ_L^N of a given fermionic system in a box of length L with and without a perturbation in the thermodynamic limit $L \rightarrow \infty$, $N \rightarrow \infty$, $N/L^d \rightarrow \text{const.} > 0$. Here, $d \in \mathbb{N}$ is the spatial dimension.

The AOC has proved to be an extremely robust phenomenon with consequences reaching far beyond this single-impurity problem, and it continues to attract attention in the physical literature. Recent studies [HSBvD05, THC⁺11, HK12a, HK12b] considered this effect in optical absorption or emission involving a single quantum-dot level hybridising with a Fermi sea. The absorption or emission of a photon induces a sudden local

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perturbation in the Fermi sea with consequences similar to that in the classical X-ray edge problem. Other manifestations in mesoscopic systems of current interest, such as graphene, can be found, e.g., in [HUB05, HG07, RH10].

P. W. Anderson was the first to explain the behaviour (1.1) in the late 1960ies. He considered a non-interacting Fermi gas in three dimensions and its perturbation by a compactly supported single-particle potential. In this setting he used Hadamard's inequality to estimate the Slater determinant of the two ground states from above – see also Lemma 3.1 – and a subsequent informal computation, which led to

$$\langle \Phi_L^N, \Psi_L^N \rangle = O(L^{-\gamma/2}) \quad (1.2)$$

in the thermodynamic limit [And67a]. Furthermore, Anderson expressed the decay exponent γ in terms of the (single-particle) scattering phases associated with the perturbation. Later on in the same year, he found a way [And67b] to circumvent Hadamard's inequality and arrived at the asymptotics (1.1) with an exponent γ bigger than that in (1.2). After some controversies about the correctness of interchanging limits, the asymptotics of [And67b] was confirmed in an adiabatic approach [RS71, Ham71, KY78]. Both this asymptotics and the bound (1.2) from [And67a] are now taken for granted in the physics literature as a fundamental property of Fermi gases.

Only very little is known about AOC from a rigorous mathematical point of view. The adiabatic approach to AOC was revisited by Otte [Ott05], who rigorously derives a limit expression for the overlap in terms of the solution of a Wiener-Hopf equation, thereby clarifying a discussion on the correctness of limits in [RS71, Ham71], see also [BC03] for related work. Unfortunately, this does not allow the thermodynamic limit to be controlled, which would be necessary for proving (1.1). Likewise, the upper bound (1.2) awaits a sound mathematical treatment. This is the goal of the present paper. We will prove (1.2) with the same decay exponent γ as in [And67a], but valid in greater generality. The recent preprint [KOS] treats the one-dimensional case, see Remark 3.8(ii) below. A related problem, recently solved in [FLLS11], is to give an effective estimate of the minimum change in total energy of the (infinite-volume) Fermi gas when a local, one-body potential is added to the kinetic energy.

The plan of this paper is as follows. We formulate our results in the next section. Sections 3 to 5 contain the proof of the main results, Theorems 2.2 and 2.2'. In the Appendix we prove Theorem 2.4 and Corollary 2.6, which relate the diagonal of the Lebesgue density of a spectral correlation measure with the scattering matrix and the cross section. This yields a scattering-theoretic interpretation of our decay exponent γ and shows that it coincides with that of [And67a].

2. Main results

We consider a pair of one-particle Schrödinger operators $H_L := -\Delta_L + V_0$ and $H'_L := H_L + V$, which are self-adjoint and densely defined in the

Hilbert space $L^2(\Lambda_L)$, where $\Lambda_L := L \cdot \Lambda_1$ is obtained from scaling some fixed bounded domain $\Lambda_1 \subseteq \mathbb{R}^d$, which contains the origin, by a factor $L > 1$. The negative Laplacian $-\Delta_L$ is supplied with Dirichlet boundary conditions on Λ_L . The background potential V_0 and the perturbation V act as multiplication operators on $L^2(\Lambda_L)$. They correspond to real-valued functions on \mathbb{R}^d (denoted by the same letter) with the properties

$$\begin{aligned} \max\{V_0, 0\} &\in K_{\text{loc}}^d(\mathbb{R}^d), \quad \max\{-V_0, 0\} \in K^d(\mathbb{R}^d), \\ V &\in L^\infty(\mathbb{R}^d), V \geq 0, \quad \text{supp}(V) \subseteq \Lambda_1 \text{ compact.} \end{aligned} \quad (\mathbf{A})$$

Here, we have written $K^d(\mathbb{R}^d)$ and $K_{\text{loc}}^d(\mathbb{R}^d)$ for the Kato class and the local Kato class, respectively [Sim82].

We denote the self-adjoint, infinite-volume operators on $L^2(\mathbb{R}^d)$, corresponding to H_L and H'_L , by H and H' . Birman's theorem [RS79, Thm. XI.10], together with [Sim82, Thm. B.9.1], guarantees the existence and completeness of the wave operators for the pair H, H' . In particular, the perturbation V does not change the absolutely continuous spectrum, i.e.

$$\sigma_{\text{ac}}(H) = \sigma_{\text{ac}}(H'). \quad (2.1)$$

Assumption **(A)** ensures that the one-particle operators H_L and H'_L in finite volume are bounded from below and have purely discrete spectrum (which follows, e.g., from the fact that the semigroup operators $\exp\{-tH_L^{(j)}\}$ are trace class [BHL00, Thm. 6.1] for each $t > 0$). We write $\lambda_1^L \leq \lambda_2^L \leq \dots$ and $\mu_1^L \leq \mu_2^L \leq \dots$ for their non-decreasing sequences of eigenvalues, counting multiplicities, and $(\varphi_j^L)_{j \in \mathbb{N}}$ and $(\psi_k^L)_{k \in \mathbb{N}}$ for the corresponding sequences of normalised eigenfunctions with an arbitrary choice of basis vectors in any eigenspace of dimension greater than one.

The induced non-interacting N -particle Schrödinger operators \mathbf{H}_L and \mathbf{H}'_L in finite volume act on the totally antisymmetric subspace $\bigwedge_{j=1}^N L^2(\Lambda_L)$ of the N -fold tensor product space and are given by

$$\mathbf{H}_L^{(j)} := \sum_{j=1}^N \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes H_L^{(j)} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}, \quad (2.2)$$

where the index j determines the position of $H_L^{(j)}$ in the N -fold tensor product of operators and $N \in \mathbb{N}$. The corresponding ground states are

$$\Phi_L^N := \frac{1}{\sqrt{N!}} \varphi_1^L \wedge \dots \wedge \varphi_N^L, \quad \Psi_L^N := \frac{1}{\sqrt{N!}} \psi_1^L \wedge \dots \wedge \psi_N^L. \quad (2.3)$$

In order to avoid ambiguities from possibly degenerate eigenspaces and to realise a given Fermi energy $E \in \mathbb{R}$ in the thermodynamic limit, we choose the number of particles as

$$N_L(E) := \#\{j \in \mathbb{N} : \lambda_j^L \leq E\} \in \mathbb{N}_0. \quad (2.4)$$

We will be interested in the ground-state overlap

$$\mathcal{S}_L(E) := \left\langle \Phi_L^{N_L(E)}, \Psi_L^{N_L(E)} \right\rangle_{N_L(E)} = \det \left(\langle \varphi_j^L, \psi_k^L \rangle \right)_{j,k=1, \dots, N_L(E)}, \quad (2.5)$$

asymptotically as $L \rightarrow \infty$. Here, $\langle \cdot, \cdot \rangle_N$ stands for the scalar product on the N -fermion space $\bigwedge_{j=1}^N L^2(\Lambda_L)$, where $N \in \mathbb{N} \setminus \{1\}$, and $\langle \cdot, \cdot \rangle$ for the one on the single-particle space $L^2(\Lambda_L)$. If $N_L(E) = 0$, we set $\mathcal{S}_L(E) := 1$.

Remark 2.1. By our choice (2.4) the particle density ϱ of the two non-interacting fermion systems in the thermodynamic limit equals the integrated density of states

$$\varrho = \lim_{L \rightarrow \infty} \frac{N_L(E)}{L^d |\Lambda_1|} \quad (2.6)$$

of the single-particle Schrödinger operator H (or equivalently that of H'), provided the limit exists. Here, $|\cdot|$ denotes the Lebesgue measure on \mathbb{R}^d . For example, the limit (2.6) exists if V_0 is periodic or vanishes at infinity. If the limit (2.6) does not exist, then there must be more than one accumulation point because $\limsup_{L \rightarrow \infty} N_L(E)/L^d < \infty$ for every $E \in \mathbb{R}$ due to assumptions (A). But even in this case it makes still sense to study the asymptotic behaviour of the overlap $\mathcal{S}_L(E)$ as $L \rightarrow \infty$.

The main result of this paper is an upper bound on the ground-state overlap $\mathcal{S}_L(E)$ for large L . Throughout we use the convention $\ln 0 := -\infty$.

Theorem 2.2. *Assume conditions (A) and let $(L_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$ be a sequence of increasing lengths with $L_n \uparrow \infty$. Then there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ and a Lebesgue null set $\mathcal{N} \subset \mathbb{R}$ of exceptional Fermi energies such that for every $E \in \mathbb{R} \setminus \mathcal{N}$ the ground-state overlap (2.5) obeys*

$$\limsup_{k \rightarrow \infty} \frac{\ln |\mathcal{S}_{L_{n_k}}(E)|}{\ln L_{n_k}} \leq -\frac{\gamma(E)}{2} \quad (2.7)$$

with some decay exponent $\gamma(E) \geq 0$.

Remarks 2.3. (i) The proof of Theorem 2.2 follows from Lemma 3.1 and Theorem 3.7 in the next section. In fact, we prove the slightly stronger statement

$$|\mathcal{S}_{L_{n_k}}(E)| \leq \exp \left[-\frac{a}{2} \gamma(E) \ln L_{n_k} + o(\ln L_{n_k}) \right] = L_{n_k}^{-a\gamma(E)/2 + o(1)} \quad (2.8)$$

as $k \rightarrow \infty$ for every $0 < a < 1$.

(ii) Of course, Theorem 2.2 is only interesting if the decay exponent $\gamma(E)$ is strictly positive. It emerges as the diagonal value of the Lebesgue density

$$\gamma(E) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mu_{\text{ac}} \left([E - \varepsilon/2, E + \varepsilon/2]^2 \right) = \left. \frac{d\mu_{\text{ac}}(E, E')}{d(E, E')} \right|_{E'=E} \quad (2.9)$$

of a spectral correlation measure, which is defined by

$$\mu_{\text{ac}}(B \times B') := \text{tr} \left(\sqrt{V} 1_B (H_{\text{ac}}) V 1_{B'} (H'_{\text{ac}}) \sqrt{V} \right), \quad B, B' \in \text{Borel}(\mathbb{R}), \quad (2.10)$$

see Definition 3.3 and Lemma 3.5 below for notations and details. In particular, we have $\gamma(E) = 0$, whenever $E \notin \sigma_{\text{ac}}(H)$. We refer to Theorem 2.4 and Corollary 2.6 for a scattering-theoretic interpretation of $\gamma(E)$.

(iii) We expect the result of Theorem 2.2 to hold true also for non-compactly supported perturbations V which decay sufficiently fast at infinity.

(iv) Anderson [And67a] treats the special case $d = 3$, $V_0 = 0$ and V spherically symmetric and argues that $\mathcal{S}_L(E) = O(L^{-\gamma(E)/2})$ as $L \rightarrow \infty$ for $E > 0$ with the same decay exponent $\gamma(E)$ as in this paper. Thus, our theorem reproduces and generalises Anderson's informal computation. We note that there is a factor of 2 missing in the final result (7) in [And67a], which was apparently forgotten.

(v) The only other mathematical work dealing with AOC is the preprint [KOS]. We refer to Remark 3.8(ii) below for a discussion.

(vi) The reason for passing to a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ in Theorem 2.2 originates from Lemma 3.9 below. What stands behind it is the lack of known a.e.-bounds on the finite-volume spectral shift function for the pair of operators H_L, H'_L , which hold uniformly in the limit $L \rightarrow \infty$. This unfortunate fact has been noticed many times in the literature, see e.g. [HM10], and the pathological behaviour of the spectral shift function found in [Kir87] illustrates that this is a delicate issue. However, in certain special situations such a.e.-bounds are known, and our result can be strengthened. More precisely, we have

Theorem 2.2'. *Assume the situation of Theorem 2.2 with $d = 1$, or replace the perturbation potential V in Theorem 2.2 by a finite-rank operator $V = \sum_{\nu=1}^n \langle \phi_\nu, \cdot \rangle \phi_\nu$ with compactly supported $\phi_\nu \in L^2(\mathbb{R}^d)$ for $\nu = 1, \dots, n$, or consider the lattice problem on \mathbb{Z}^d corresponding to the situation in Theorem 2.2. Then the ground-state overlap (2.5) obeys*

$$\limsup_{L \rightarrow \infty} \frac{\ln |\mathcal{S}_L(E)|}{\ln L} \leq -\frac{\gamma(E)}{2} \quad (2.11)$$

with some decay exponent $\gamma(E) \geq 0$ for Lebesgue-a.e. $E \in \mathbb{R}$.

Next we turn to the already mentioned interpretation of the decay exponent in terms of quantities from scattering theory. Such a relation between the density of a spectral correlation measure and the scattering matrix or cross section, which we have not found in the literature in this particular form, may be of independent interest. In our case, this relation reveals non-trivial scattering as a mechanism leading to AOC.

Theorem 2.4. *Assume (A) with $V_0 = 0$. Then the decay exponent $\gamma(E)$ in Theorem 2.2 reflects the amount of scattering caused by the perturbation and is given by*

$$\gamma(E) = \frac{E^{(d-1)/2}}{(2\pi)^{d+1}} \int_{\mathbb{S}^{d-1}} d\Omega(\omega) \sigma(E, \omega) = (2\pi)^{-2} \|S(E) - \mathbf{1}\|_{\text{HS}}^2 \quad (2.12)$$

for Lebesgue-a.e. $E \geq 0$ and $\gamma(E) = 0$ for $E < 0$. Here, $\sigma(E, \omega)$ stands for the total scattering cross-section for the pair of operators H, H' on the energy shell corresponding to E with incident direction $\omega \in \mathbb{S}^{d-1}$ and $d\Omega$ is the Lebesgue measure on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d . On the right-hand side

$S(E) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ denotes the scattering matrix and $\|\cdot\|_{\text{HS}}$ the Hilbert-Schmidt norm.

Remarks 2.5. (i) The theorem will be proven in the Appendix using generalised eigenfunctions.

(ii) We refer to [Yaf00] for precise definitions of the scattering-theoretic quantities. We suspect that the theorem remains true for general Kato decomposable background potentials V_0 , and also under the conditions of Theorem 2.2'. In fact, the relations in [BE67, §7], which do not rely on generalised eigenfunctions, seem to indicate this.

In order to see that our findings agree with those of Anderson [And67a], we further specialise to $d = 3$ dimensions and a spherically symmetric perturbation V .

Corollary 2.6. *Let $d = 3$. Assume (A) with $V_0 = 0$ and V spherically symmetric. Then the decay exponent $\gamma(E)$ in Theorem 2.2 is given by*

$$\gamma(E) = \frac{1}{\pi^2} \sum_{\ell=0}^{\infty} (2\ell + 1) (\sin \delta_{\ell}(E))^2 \quad (2.13)$$

for Lebesgue-a.e. $E \geq 0$ and $\gamma(E) = 0$ for $E < 0$. Here, $\delta_{\ell}(E)$, $\ell \in \mathbb{N}_0$, are the scattering phases.

Remarks 2.7. (i) The proof of Corollary 2.6 will also be given in the Appendix.

(ii) We refer to [RS79, XI.8.C] for a definition of the scattering phases.

3. Proof of Theorems 2.2 and 2.2'

We start by estimating the ground-state overlap in the same way as in the first step of [And67a].

Lemma 3.1. *For every length $L > 0$ and every Fermi energy $E \in \mathbb{R}$ we define the Anderson integral*

$$\mathcal{I}_L(E) := \sum_{j=1}^{N_L(E)} \sum_{k=N_L(E)+1}^{\infty} |\langle \varphi_j^L, \psi_k^L \rangle|^2 \quad (3.1)$$

and obtain the estimate

$$|\mathcal{S}_L(E)| \leq \exp[-\tfrac{1}{2}\mathcal{I}_L(E)]. \quad (3.2)$$

PROOF. The assertion is true by definition if $N_L(E) = 0$. For $N_L(E) \geq 1$, we use Hadamard's inequality for determinants to estimate

$$|\mathcal{S}_L(E)| = \left| \det(\langle \varphi_j^L, \psi_k^L \rangle)_{j,k=1,\dots,N_L(E)} \right| \leq \prod_{j=1}^{N_L(E)} \left(\sum_{k=1}^{N_L(E)} |\langle \varphi_j^L, \psi_k^L \rangle|^2 \right)^{\frac{1}{2}} \quad (3.3)$$

and therefore

$$\ln|\mathcal{S}_L(E)| \leq \frac{1}{2} \sum_{j=1}^{N_L(E)} \ln \left(\sum_{k=1}^{N_L(E)} |\langle \varphi_j^L, \psi_k^L \rangle|^2 \right). \quad (3.4)$$

Parseval's identity $\sum_{k=1}^{N_L(E)} |\langle \varphi_j^L, \psi_k^L \rangle|^2 = 1 - \sum_{k=N_L(E)+1}^{\infty} |\langle \varphi_j^L, \psi_k^L \rangle|^2$ and the elementary inequality $\ln(1+x) \leq x$ for $x \geq -1$ then yield the claim of the lemma. \square

In order to define the decay exponent $\gamma(E)$ of the main theorem, we need a convergence result due to Birman and Èntina. We write $H_{ac}^{(\prime)}$ to denote the restriction of the operator $H^{(\prime)}$ to its absolutely continuous subspace, and 1_B stands for the indicator function of a set B .

Proposition 3.2 ([BÈ67, Lemma 4.3]). *Assume the situation of Theorem 2.2 or Theorem 2.2'. For $E \in \mathbb{R}$ and $\varepsilon > 0$ we define the spectral projections*

$$P_E^\varepsilon := 1_{]E-\varepsilon/2, E+\varepsilon/2[}(H_{ac}), \quad \Pi_E^\varepsilon := 1_{]E-\varepsilon/2, E+\varepsilon/2[}(H_{ac}'). \quad (3.5)$$

Then there exists a Lebesgue null set $\mathcal{N}_0 \subset \mathbb{R}$ such that the limits

$$P_E := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sqrt{V} P_E^\varepsilon \sqrt{V}, \quad \Pi_E := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sqrt{V} \Pi_E^\varepsilon \sqrt{V} \quad (3.6)$$

exist in trace class for all $E \in \mathbb{R} \setminus \mathcal{N}_0$ and define nonnegative trace class operators P_E and Π_E .

The above proposition guarantees that the quantities introduced in the first part of the next definition are well-defined.

Definition 3.3. (i) For $E, E' \in \mathbb{R} \setminus \mathcal{N}_0$ we introduce

$$\gamma_1(E) := \text{tr } P_E, \quad \gamma_2(E) := \text{tr } \Pi_E \quad (3.7)$$

as well as the two-dimensonal quantity

$$\gamma^{(2)}(E, E') := \text{tr}(P_E \Pi_{E'}) \quad (3.8)$$

and its value on the diagonal

$$\gamma(E) := \gamma^{(2)}(E, E). \quad (3.9)$$

This gives rise to functions $\gamma_1, \gamma_2, \gamma : \mathbb{R} \rightarrow \mathbb{R}$ and $\gamma^{(2)} : \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting them to zero if the limits in (3.6) do not exist.

(ii) The Borel measures μ_{ac}^1 and μ_{ac}^2 on \mathbb{R} are defined by

$$\mu_{ac}^1(B) := \text{tr}(\sqrt{V} 1_B(H_{ac}) \sqrt{V}), \quad \mu_{ac}^2(B) := \text{tr}(\sqrt{V} 1_B(H_{ac}') \sqrt{V}) \quad (3.10)$$

for $B \in \text{Borel}(\mathbb{R})$.

(iii) The spectral correlation measure μ_{ac} on \mathbb{R}^2 is defined by

$$\mu_{ac}(B \times B') := \text{tr} \left(\sqrt{V} 1_B(H_{ac}) V 1_{B'}(H_{ac}') \sqrt{V} \right) \quad (3.11)$$

for $B, B' \in \text{Borel}(\mathbb{R})$.

Remark 3.4. Both expressions in (3.10) define Borel measures by [Sim82, Thm. B.9.2]. They are absolutely continuous with respect to Lebesgue measure on \mathbb{R} and the functions γ_1 , resp. γ_2 are representatives of the Lebesgue densities of $\mu_{\text{ac}}^{(1)}$, resp. $\mu_{\text{ac}}^{(2)}$. In particular, $\gamma_1, \gamma_2 \in L^1_{\text{loc}}(\mathbb{R})$.

A corresponding statement for the two-dimensional measure μ_{ac} is contained in

Lemma 3.5. *The Borel measure μ_{ac} is well-defined and absolutely continuous with respect to Lebesgue measure on \mathbb{R}^2 . The function $\gamma^{(2)} \in L^1_{\text{loc}}(\mathbb{R}^2)$ is a representative of its Lebesgue density and obeys*

$$\gamma^{(2)}(E, E') \leq \gamma_1(E)\gamma_2(E') \quad \text{for all } E, E' \in \mathbb{R}. \quad (3.12)$$

PROOF OF LEMMA 3.5. Let $B, B' \in \text{Borel}(\mathbb{R})$ be bounded. Then the non-negative expression (3.11) is finite because of [Sim82, Thm. B.9.1]. Thus, it gives rise to a uniquely defined Borel measure on \mathbb{R}^2 by standard reasoning [Bau01, Thm. 23.3].

Hölder's inequality and the norm inequality $\|\cdot\|_{\ell^2} \leq \|\cdot\|_{\ell^1}$ for the (standard) sequence spaces imply the estimate

$$\text{tr} \left(\sqrt{V} 1_B (H_{\text{ac}}) V 1_{B'} (H'_{\text{ac}}) \sqrt{V} \right) \leq \text{tr} \left(\sqrt{V} 1_B (H_{\text{ac}}) \sqrt{V} \right) \text{tr} \left(\sqrt{V} 1_{B'} (H'_{\text{ac}}) \sqrt{V} \right). \quad (3.13)$$

The inequality (3.12) follows directly from it. In turn, (3.12) and Remark 3.4 imply $\gamma^{(2)} \in L^1_{\text{loc}}(\mathbb{R}^2)$.

To show absolute continuity of μ_{ac} , we conclude from (3.13) and Remark 3.4 that

$$\mu_{\text{ac}}(C) \leq \int_C dE dE' \gamma_1(E) \gamma_2(E') \quad (3.14)$$

holds for all product sets $C = B \times B'$ with $B, B' \in \text{Borel}(\mathbb{R})$. The comparison theorem [Els05, Thm. II.5.8] extends (3.14) to all $C \in \text{Borel}(\mathbb{R}^2)$. In particular, μ_{ac} is absolutely continuous with respect to two-dimensional Lebesgue measure.

Due to absolute continuity of μ_{ac} the limit

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mu_{\text{ac}} \left([E - \varepsilon/2, E + \varepsilon/2] \times [E' - \varepsilon/2, E' + \varepsilon/2] \right) = \frac{d\mu_{\text{ac}}(E, E')}{d(E, E')} \quad (3.15)$$

exists for Lebesgue-a.e. $(E, E') \in \mathbb{R}^2$. But, by definition, the left-hand side equals $\gamma^{(2)}(E, E')$ for all $E, E' \in \mathbb{R} \setminus \mathcal{N}_0$. \square

Remark 3.6. We work with a particular representative of the Lebesgue density of μ_{ac} because we are interested in diagonal values $\gamma^{(2)}(E, E) = \gamma(E)$ of the density.

Theorems 2.2 and 2.2' will follow from Lemma 3.1 and

Theorem 3.7. (i) *Assume conditions (A) and let $(L_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$ be a sequence of increasing lengths with $L_n \uparrow \infty$. Then there exists a subsequence*

$(L_{n_k})_{k \in \mathbb{N}}$ and a Lebesgue null set $\mathcal{N} \subset \mathbb{R}$ of exceptional Fermi energies such that for every $E \in \mathbb{R} \setminus \mathcal{N}$ the Anderson integral (3.1) obeys

$$\mathcal{I}_{L_{n_k}}(E) \geq a\gamma(E) \ln L_{n_k} + o(\ln L_{n_k}) \quad \text{as } k \rightarrow \infty \quad (3.16)$$

for every $0 < a < 1$ and with $\gamma(E)$ given by (3.9).

(ii) Assume the situation of Theorem 2.2'. Then there exists a Lebesgue null set $\mathcal{N} \subset \mathbb{R}$ of exceptional Fermi energies, such that for every $E \in \mathbb{R} \setminus \mathcal{N}$ the Anderson integral (3.1) obeys

$$\mathcal{I}_L(E) \geq a\gamma(E) \ln L + o(\ln L) \quad \text{as } L \rightarrow \infty \quad (3.17)$$

for every $0 < a < 1$ and with $\gamma(E)$ given by (3.9).

Remarks 3.8. (i) Theorem 3.7 follows immediately from Lemma 3.9 and Theorem 3.10.

(ii) The only other mathematical investigation of AOC we are aware of is the recent preprint [KOS]. There the special case $d = 1$ and $V_0 = 0$ is treated. Moreover, the perturbation V needs to be small in a certain sense. But it is not restricted to compact support – sufficiently fast decay is enough. The result of [KOS] is an asymptotic evaluation of the Anderson integral \mathcal{I}_L as $L \rightarrow \infty$. Technically, it relies on the exact knowledge of the eigenvalues and eigenfunctions of the one-dimensional Laplacian in an interval and sophisticated explicit computations. The so-obtained decay exponent still awaits an interpretation in terms of quantities from scattering theory. For this reason it is currently not clear whether their decay exponent agrees with Anderson's and our γ .

The next lemma estimates the error arising from a modification of the Anderson integral so that all energy levels up to, respectively from, the *same* energy are taken into account. This is where the spectral shift function enters. It is only part (i) of this lemma which forces us to pick a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ of the original sequence of lengths $(L_n)_{n \in \mathbb{N}}$.

Lemma 3.9. (i) Assume (A) and let $(L_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{\geq 0}$ be a sequence of increasing lengths with $L_n \uparrow \infty$. Then there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ such that for Lebesgue-a.e. Fermi energies $E \in \mathbb{R}$

$$\left| \mathcal{F}_{L_{n_k}}(E) - \mathcal{I}_{L_{n_k}}(E) \right| = o(\ln L_{n_k}) \quad (3.18)$$

as $k \rightarrow \infty$. Here,

$$\mathcal{F}_L(E) := \text{tr} \left(1_{]-\infty, E]}(H) 1_{]E, \infty[}(H') \right) \quad (3.19)$$

is the fixed-energy Anderson integral.

(ii) Assume the situation of Theorem 2.2'. Then

$$\limsup_{L \rightarrow \infty} \left| \mathcal{F}_L(E) - \mathcal{I}_L(E) \right| < \infty, \quad (3.20)$$

for every Fermi energy $E \in \mathbb{R}$.

PROOF. Given $L > 0$ and $E \in \mathbb{R}$, we recall from (2.4) that, by definition,

$$\lambda_{N_L(E)}^L \leq E \quad \text{and} \quad \lambda_{N_L(E)+1}^L > E, \quad (3.21)$$

where we use the convention $\lambda_0^L := -\infty$. This allows to rewrite the Anderson integral as

$$\begin{aligned} \mathcal{I}_L(E) &= \sum_{j=1}^{N_L(E)} \sum_{k=N_L(E)+1}^{\infty} |\langle \varphi_j^L, \psi_k^L \rangle|^2 \\ &= \text{tr} \left\{ 1_{]-\infty, E]}(H) \left(1_{]E, \infty[}(H') - \sum_{k \in \{1, \dots, N_L(E)\} : \mu_k^L > E} \langle \psi_k^L, \cdot \rangle \psi_k^L \right) \right\}. \end{aligned} \quad (3.22)$$

The number of terms in the above k -sum

$$\#\{k \in \{1, \dots, N_L(E)\} : \mu_k^L > E\} = N_L(E) - \#\{k \in \mathbb{N} : \mu_k^L \leq E\} =: \xi_L(E) \quad (3.23)$$

is precisely the value at E of the (non-negative) spectral shift function for the pair of finite-volume operators H_L, H'_L . Therefore we obtain

$$0 \leq \mathcal{F}_L(E) - \mathcal{I}_L(E) \leq \xi_L(E), \quad (3.24)$$

and it remains to prove that this error is of order $o(\ln L)$ as $L \rightarrow \infty$. In the situation of (ii), we have even $\sup_{L \geq 1} \xi_L(E) < \infty$ for every $E \in \mathbb{R}$ thanks to a finite-rank argument and the min-max principle. In order to apply this finite-rank argument in the one-dimensional continuum case, use Dirichlet-Neumann bracketing and the fact that introducing a Dirichlet or Neumann boundary point amounts to a rank-two-perturbation.

In the multi-dimensional continuum situation of (i) no such a.e.-bounds are known. But we can exploit the weak convergence [HM10, Thm. 1.4]

$$\lim_{L \rightarrow \infty} \int_I dE \xi_L(E) = \int_I dE \xi(E), \quad (3.25)$$

for every bounded interval $I \subseteq \mathbb{R}$, where $\xi \in L^1_{\text{loc}}(\mathbb{R})$ is the spectral shift function for the pair of infinite-volume operators H, H' . Thus, given a sequence of diverging lengths $(L_n)_{n \in \mathbb{N}}$, the sequence of non-negative functions $(\xi_{L_n} / \ln L_n)_{n \in \mathbb{N}}$ converges to zero in $L^1(I)$. Hence there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ such that $(\xi_{L_{n_k}} / \ln L_{n_k})_{k \in \mathbb{N}}$ converges to zero for Lebesgue-a.e. $E \in I$. The claim then follows from exhausting \mathbb{R} by a sequence of bounded intervals I . \square

Theorem 3.10. *Assume the situation of Theorem 2.2 or Theorem 2.2'. Then there exists a Lebesgue null set $\mathcal{N} \subset \mathbb{R}$ of exceptional Fermi energies such that for every $E \in \mathbb{R} \setminus \mathcal{N}$ and every $a \in]0, 1[$*

$$\mathcal{F}_L(E) \geq a\gamma(E) \ln L + o(\ln L) \quad (3.26)$$

as $L \rightarrow \infty$.

We will explicitly spell out the proof of Theorem 3.10 for the situation of Theorem 2.2 only. The proof is fully analogous (and even simpler) in the remaining situations of Theorem 2.2', where V is a finite-rank operator or that of the lattice model.

In the first lemma which enters the proof of Theorem 3.10 we rewrite the fixed-energy Anderson integral as an integral with respect to a spectral correlation measure.

Lemma 3.11. *Assume (A), let $L > 0$ and $E \in \mathbb{R}$. Then we have*

$$\begin{aligned} \mathcal{F}_L(E) &= \int_{]-\infty, E] \times]E, \infty[} \frac{d\mu_L(x, y)}{(y - x)^2} \\ &\geq \int_0^{L^a} dt \, t \int_{\mathbb{R}^2} d\mu_L(x, y) e^{-t(y-x)} \chi_L^-(x) \chi_L^+(y), \end{aligned} \quad (3.27)$$

where the (finite-volume) spectral correlation measure μ_L on \mathbb{R}^2 is uniquely defined by $\mu_L(B \times B') := \text{tr}(\sqrt{V} 1_B(H) V 1_{B'}(H') \sqrt{V})$ for $B, B' \in \text{Borel}(\mathbb{R})$. The parameter $a > 0$ and the functions $\chi_L^\pm \in L^\infty(\mathbb{R})$ are arbitrary subject to

$$0 \leq \chi_L^+ \leq 1_{]E, \infty[} \quad \text{and} \quad 0 \leq \chi_L^- \leq 1_{]-\infty, E]}. \quad (3.28)$$

Remark 3.12. We have suppressed the dependence of χ_L^\pm on the Fermi energy E and will impose further properties on these functions in Definition 3.13 below.

PROOF OF LEMMA 3.11. The eigenvalue equations imply

$$\lambda_j^L \langle \varphi_j^L, \psi_k^L \rangle = \langle H_L \varphi_j^L, \psi_k^L \rangle = \mu_k^L \langle \varphi_j^L, \psi_k^L \rangle - \langle \varphi_j^L, V \psi_k^L \rangle \quad (3.29)$$

from which we obtain the identity

$$|\langle \varphi_j^L, \psi_k^L \rangle|^2 = \frac{|\langle \varphi_j^L, V \psi_k^L \rangle|^2}{(\mu_k^L - \lambda_j^L)^2}, \quad (3.30)$$

provided $\lambda_j^L \neq \mu_k^L$. This yields

$$\begin{aligned} \mathcal{F}_L(E) &= \sum_{\substack{j \in \mathbb{N}: \\ \lambda_j^L \leq E}} \sum_{\substack{k \in \mathbb{N}: \\ \mu_k^L > E}} |\langle \varphi_j^L, \psi_k^L \rangle|^2 = \sum_{\substack{j \in \mathbb{N}: \\ \lambda_j^L \leq E}} \sum_{\substack{k \in \mathbb{N}: \\ \mu_k^L > E}} \frac{|\langle \varphi_j^L, V \psi_k^L \rangle|^2}{(\mu_k^L - \lambda_j^L)^2} \\ &= \int_{]-\infty, E] \times]E, \infty[} \frac{d\mu_L(x, y)}{(y - x)^2}. \end{aligned} \quad (3.31)$$

The inequality in (3.27) follows from the integral representation $x^{-2} = \int_0^\infty dt \, t e^{-tx}$ for $x > 0$, Fubini's theorem, from cutting the t -integral and (3.28). \square

Definition 3.13. Given an exponent $b > 0$, a length $L > 1$, a cut-off energy $E_0 \geq 1$ and a Fermi energy $E \in [-E_0 + 1, E_0 - 1]$, we say that $\chi_L^\pm \in C_c^\infty(\mathbb{R})$

are *smooth cut-off functions*, if they obey

$$\begin{aligned} 1_{[E+L^{-b}, E_0]} &\leq \chi_L^+ \leq 1_{]E, E_0+1[}, \\ 1_{[-E_0, E-L^{-b}]} &\leq \chi_L^- \leq 1_{]-E_0-1, E[} \end{aligned} \quad (3.32)$$

and if there exist L -independent constants $c_\nu \in \mathbb{R}_{>0}$, $\nu \in \mathbb{N}_0$, such that

$$\chi_L^\pm(E \pm x) \leq c_0 L^b x \quad (3.33)$$

for all $x \in [0, L^{-b}[$ and

$$\left| \frac{\partial^\nu}{\partial x^\nu} \chi_L^\pm(E \pm x) \right| \leq \begin{cases} c_\nu L^{\nu b} & \text{for all } x \in [0, L^{-b}[, \\ c_\nu & \text{otherwise} \end{cases} \quad (3.34)$$

for every $\nu \in \mathbb{N}$.

Thus, χ_L^+ equals one inside $[E+L^{-b}, E_0]$ and zero in $] -\infty, E] \cup [E_0+1, \infty[$. Whereas its smooth growth in $[E, E+L^{-b}]$ gets steeper with increasing L , we choose its smooth decay in $[E_0, E_0+1]$ independently of L . The properties of χ_L^- are analogous.

The next lemma allows to replace the finite-volume operators by their infinite-volume analogues. It is the crucial step in our argument, and we defer the proof to Section 4.

Lemma 3.14. *Let $0 < a < b < 1$, $L > 1$ and $E_0 \geq 1$. Pick a Fermi energy $E \in [-E_0+1, E_0-1]$ and let χ_L^\pm be the associated smooth cut-off functions. Then we have*

$$\begin{aligned} \int_0^{L^a} dt \, t \left(\int_{\mathbb{R}^2} d\mu_L(x, y) e^{-t(y-x)} \chi_L^-(x) \chi_L^+(y) \right. \\ \left. - \int_{\mathbb{R}^2} d\mu(x, y) e^{-t(y-x)} \chi_L^-(x) \chi_L^+(y) \right) = o(1) \end{aligned} \quad (3.35)$$

as $L \rightarrow \infty$, where the (infinite-volume) spectral correlation measure μ on \mathbb{R}^2 is uniquely defined by

$$\mu(B \times B') := \text{tr} \left(\sqrt{V} 1_B(H) V 1_{B'}(H') \sqrt{V} \right) \quad (3.36)$$

for $B, B' \in \text{Borel}(\mathbb{R})$.

We recall the measure μ_{ac} from Definition 3.3 and the Lebesgue densities γ_1 and γ_2 of the measures $\mu_{\text{ac}}^{(1)}$ and $\mu_{\text{ac}}^{(2)}$, see Remark 3.4. In the next lemma we estimate the error for replacing the smooth cut-off functions in the limit expression (more precisely, its ac-part) of the previous lemma by step functions.

Lemma 3.15. *In addition to the hypotheses of the previous lemma, suppose that $E \in [-E_0+1, E_0-1]$ is a Lebesgue point of both γ_1 and γ_2 . Then we have*

$$\int_0^{L^a} dt \, t \int_{\mathbb{R}^2} d\mu_{\text{ac}}(x, y) e^{-t(y-x)} \left(\chi_L^-(x) \chi_L^+(y) - 1_{[-E_0, E]}(x) 1_{[E, E_0]}(y) \right) = O(1) \quad (3.37)$$

as $L \rightarrow \infty$.

The proof of this lemma is deferred to Section 5.

In the last lemma, we show how the diagonal of the μ_{ac} -density arises in the large- t limit.

Lemma 3.16. *For Lebesgue-a.e. $E \in [-E_0, E_0]$ we have*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^2} d\mu_{\text{ac}}(x, y) t^2 e^{-t(y-x)} 1_{[-E_0, E]}(x) 1_{[E, E_0]}(y) = \gamma(E). \quad (3.38)$$

PROOF. To shorten formulas, we suppress the subscript ac and write $H^{(t)} = H_{\text{ac}}^{(t)}$ in this proof. Recalling the definition of μ_{ac} and the identity $\text{tr}(P_E \Pi_E) = \gamma(E)$, which is valid for Lebesgue-a.e. $E \in \mathbb{R}$, we have to show that

$$\left| \text{tr} \left\{ \sqrt{V} t e^{t(H-E)} 1_{[-E_0, E]}(H) V t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \right\} - \text{tr}(P_E \Pi_E) \right| \quad (3.39)$$

vanishes as $t \rightarrow \infty$. We bound this expression from above by

$$\begin{aligned} & \left| \text{tr} \left\{ \left(\sqrt{V} t e^{t(H-E)} 1_{[-E_0, E]}(H) \sqrt{V} - P_E \right) \sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \right\} \right. \\ & \quad \left. + \text{tr} \left\{ P_E \left(\sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} - \Pi_E \right) \right\} \right| \\ & \leq \left(\sup_{t \geq 0} \left\| \sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \right\| \right) \\ & \quad \times \left| \text{tr} \left(\sqrt{V} t e^{t(H-E)} 1_{[-E_0, E]}(H) \sqrt{V} - P_E \right) \right| \\ & \quad + \|P_E\| \left| \text{tr} \left(\sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} - \Pi_E \right) \right|. \end{aligned} \quad (3.40)$$

First, we claim that

$$\lim_{t \rightarrow \infty} \sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} = \Pi_E \quad (3.41)$$

in trace class, for Lebesgue-a.e. $E \in [-E_0, E_0]$. To see this, we show convergence of the trace norms and weak convergence, which implies convergence in trace class by [Sim05, Addendum H]. For the trace norms, we compute

$$\begin{aligned} \text{tr} \left| \sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \right| &= \text{tr} \left(\sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \right) \\ &= \int_E^{E_0} dy \gamma_2(y) t e^{-t(y-E)} \\ &= ((\gamma_2 1_{[-E_0, E_0]}) * \varrho_t)(E), \end{aligned} \quad (3.42)$$

with $x \mapsto \varrho_t(x) := t e^{tx} 1_{-\infty, 0]}(x)$ being an approximation of the Dirac delta distribution. As $t \rightarrow \infty$, the convolution in (3.42) converges for Lebesgue-a.e. $E \in [-E_0, E_0]$ to $\gamma_2(E) = \text{tr} \Pi_E = \text{tr} |\Pi_E|$, see e.g. [Ste93, Chap. 1]. Thus, the trace norm of $\sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V}$ converges to that of Π_E . In particular,

$$\sup_{t \geq 0} \left\| \sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \right\| < \infty. \quad (3.43)$$

It remains to show weak convergence. To this end, take some dense countable set $\mathcal{D} \subseteq L^2(\mathbb{R}^d)$. Then by a similar delta-argument as above

$$\lim_{t \rightarrow \infty} \langle \varphi, \sqrt{V} t e^{-t(H'-E)} 1_{[E, E_0]}(H') \sqrt{V} \psi \rangle = \langle \varphi, \Pi_E \psi \rangle \quad (3.44)$$

for all $\varphi, \psi \in \mathcal{D}$ and all $E \in [-E_0, E_0]$ outside a null set depending on \mathcal{D} . Together with (3.43), this proves weak convergence to Π_E for Lebesgue-a.e. $E \in [-E_0, E_0]$, see [Wei80, Theorem 4.26].

The same argument proves $\lim_{t \rightarrow \infty} \sqrt{V} t e^{t(H-E)} 1_{[-E_0, E]}(H) \sqrt{V} = P_E$ in trace class. Using this, (3.43), the boundedness of P_E and (3.41), the right-hand side of (3.40) is seen to vanish as $t \rightarrow \infty$. \square

We are now ready for the proof of Theorem 3.10, which also completes the proof of Theorem 3.7 and, thus, of Theorems 2.2 and 2.2'.

PROOF OF THEOREM 3.10. Let $0 < a < b < 1$. Lemmas 3.11, 3.14 and 3.15 imply that

$$\mathcal{F}_L(E) \geq \int_0^{L^a} dt \, t \int_{[-E_0, E] \times [E, E_0]} d\mu_{\text{ac}}(x, y) e^{-t(y-x)} + O(1) \quad (3.45)$$

as $L \rightarrow \infty$ for Lebesgue-a.e. $E \in [-E_0 + 1, E_0 - 1]$ (more precisely those E which are Lebesgue points of γ_1 and γ_2). Here, we have also used $\mu(C) \geq \mu_{\text{ac}}(C)$ for every $C \in \text{Borel}(\mathbb{R}^2)$ and the non-negativity of the integrand.

Furthermore, using Lemma 3.16,

$$\lim_{L \rightarrow \infty} \frac{1}{\ln L} \int_1^{L^a} \frac{dt}{t} \left| \int_{[-E_0, E] \times [E, E_0]} d\mu_{\text{ac}}(x, y) \, t^2 e^{-t(y-x)} - \gamma(E) \right| = 0 \quad (3.46)$$

for Lebesgue-a.e. $E \in [-E_0 + 1, E_0 - 1]$ (the exceptional set being independent of a). Thus, we can replace the integrand in (3.45) by $\gamma(E)$ at the expense of a sublogarithmic error and arrive at

$$\mathcal{F}_L(E) \geq \int_1^{L^a} \frac{dt}{t} \gamma(E) + o(\ln L) \quad (3.47)$$

as $L \rightarrow \infty$ for Lebesgue-a.e. $E \in \mathbb{R}$, which proves the theorem. \square

4. Proof of Lemma 3.14

To shorten the formulas, we assume w.l.o.g. that $E = 0$. This can always be achieved by an energy shift of the Hamiltonians. We define the abbreviations

$$g_L^t(x) := \chi_L^-(x) e^{tx} \quad \text{and} \quad f_L^t(x) := \chi_L^+(x) e^{-tx} \quad (4.1)$$

for every $x \in \mathbb{R}$ and $t \geq 0$ so that Lemma 3.14 can be reformulated as

$$\int_0^{L^a} dt \, t K_L(t) = o(1) \quad (4.2)$$

as $L \rightarrow \infty$ with

$$K_L(t) := \text{tr} \left(\sqrt{V} g_L^t(H_L) V f_L^t(H'_L) \sqrt{V} \right) - \text{tr} \left(\sqrt{V} g_L^t(H) V f_L^t(H') \sqrt{V} \right). \quad (4.3)$$

We estimate this function according to $|K_L(t)| \leq K_L^{(1)}(t) + K_L^{(2)}(t)$, where

$$K_L^{(1)}(t) := \operatorname{tr} \left(\sqrt{V} f_L^t(H') \sqrt{V} \right) \left\| \sqrt{V} \left(g_L^t(H_L) - g_L^t(H) \right) \sqrt{V} \right\|, \quad (4.4)$$

$$K_L^{(2)}(t) := \operatorname{tr} \left(\sqrt{V} g_L^t(H_L) \sqrt{V} \right) \left\| \sqrt{V} \left(f_L^t(H_L) - f_L^t(H') \right) \sqrt{V} \right\|. \quad (4.5)$$

Since both $K_L^{(1)}$ and $K_L^{(2)}$ can be estimated in the very same way, we will demonstrate the argument for $K_L^{(2)}$ only. Our main technical tool is the Helffer-Sjöstrand formula, see e.g. [HS00, Section IX], according to which

$$f_L^t(H_L) - f_L^t(H') = \frac{1}{2\pi} \int_{\mathbb{C}} dz \, (\partial_{\bar{z}} \tilde{f}_L^t(z)) \left(\frac{1}{H_L' - z} - \frac{1}{H' - z} \right). \quad (4.6)$$

Here, $z := x + iy$, $\partial_{\bar{z}} := \partial_x + i\partial_y$, $dz := dx dy$ and $\tilde{f}_L^t \in C_c^2(\mathbb{C})$ is an almost analytic extension of f_L^t to the complex plane. The latter can be chosen as

$$\tilde{f}_L^t(z) := \xi(z) \sum_{k=0}^n \frac{(iy)^k}{k!} \frac{d^k f_L^t}{dx^k}(x) \quad (4.7)$$

for some $n \in \mathbb{N}$ and some $\xi \in C_c^\infty(\mathbb{C})$ with $\xi(z) = 1$ for all $z \in \operatorname{supp} f_L^t \times [-1, 1]$, $\xi(z) = 0$ for all z such that $\operatorname{dist}_{\mathbb{C}}(z, \operatorname{supp} f_L^t) \geq 3$ and $\xi(z) \in [0, 1]$ otherwise. We will assume $n \geq 2$ below. Since $\operatorname{supp} f_L^t = [0, E_0 + 1]$, the function ξ can be chosen independently of L and t , and such that $\|\xi\|_\infty = 1$ and $\|\xi'\|_\infty < 1$.

For later purpose we introduce the function $h := \sum_{k=0}^{n+1} \left| \frac{d^k f_L^t}{dx^k} \right| \in C_c^\infty(\mathbb{R})$ and infer the existence of a constant $C \in]0, \infty[$, which is independent of L and t , such that

$$|\partial_{\bar{z}} \tilde{f}_L^t(z)| \leq C |y|^n h(x) \quad (4.8)$$

for all $z \in \mathbb{C}$. Furthermore, the bound (3.34) implies the estimate

$$\left| \frac{d^k f_L^t}{dx^k}(x) \right| \leq L^{bk} \sum_{\kappa=0}^k \binom{k}{\kappa} \left(\frac{t}{L^b} \right)^\kappa c_{k-\kappa} 1_{[0, E_0+1]}(x) \quad (4.9)$$

for every $t \geq 0$, $L \geq 1$ and $x \in \mathbb{R}$. From this we conclude the existence of a polynomial Q_n over \mathbb{R} of degree $n+1$ with non-negative coefficients such that

$$0 \leq h(x) \leq Q_n(t/L^b) L^{b(n+1)} 1_{[0, E_0+1]}(x). \quad (4.10)$$

We will split the contribution of (4.6) in (4.5) into two parts. Accordingly, we define for $\varepsilon \in]0, 1 - b[$

$$D_L^<(t) := \frac{1}{2\pi} \int_{|y| \leq L^{-1+\varepsilon}} dz \, (\partial_{\bar{z}} \tilde{f}_L^t(z)) \sqrt{V} \left[\frac{1}{H_L' - z} - \frac{1}{H' - z} \right] \sqrt{V} \quad (4.11)$$

and

$$D_L^>(t) := \frac{1}{2\pi} \int_{|y| > L^{-1+\varepsilon}} dz \, (\partial_{\bar{z}} \tilde{f}_L^t(z)) \sqrt{V} \left[\frac{1}{H_L' - z} - \frac{1}{H' - z} \right] \sqrt{V}. \quad (4.12)$$

Then, using the boundedness of V and the estimates (4.8) and (4.10), we obtain

$$\begin{aligned}
\|D_L^<(t)\| &\leq \frac{1}{2\pi} \int_{|y| \leq L^{-1+\varepsilon}} dz |\partial_{\bar{z}} \tilde{f}(z)| \frac{2}{|y|} \|\sqrt{V}\|^2 \\
&\leq \frac{C}{\pi} \|V\|_\infty \int_{|y| \leq L^{-1+\varepsilon}} dz |y|^{n-1} h(x) \\
&= \frac{2C}{\pi n} \|V\|_\infty L^{n(-1+\varepsilon)} \int_{\mathbb{R}} dx h(x) \\
&\leq C_< Q_n(t/L^b) L^{b+n(-1+\varepsilon+b)}, \tag{4.13}
\end{aligned}$$

where $C_< := (2C/\pi n)(E_0 + 1)\|V\|_\infty < \infty$.

We estimate the norm of (4.12) with the help of the geometric resolvent equation – see e.g. [Sto01, Lemma 2.5.2], whose proof extends to Kato decomposable potentials – the bound (4.8) and the fact that $\xi(z) = 0$ if $\text{dist}(z, \mathbb{R}) \geq 3$. This gives for $L > 3$

$$\begin{aligned}
\|D_L^>(t)\| &\leq \frac{C_{\text{GRE}}}{2\pi} \|V\|_\infty \int_{|y| > L^{-1+\varepsilon}} dz |\partial_{\bar{z}} \tilde{f}(z)| \left\| 1_{\text{supp } V} \frac{1}{H'_L - z} 1_{\delta\Lambda_L} \right\| \\
&\quad \times \left\| 1_{\delta\Lambda_L} \frac{1}{H' - z} 1_{\text{supp } V} \right\| \\
&\leq \frac{CC_{\text{GRE}}}{2\pi} \|V\|_\infty \int_{|y| \in]L^{-1+\varepsilon}, 3]} dz h(x) |y|^{n-1} \left\| 1_{\delta\Lambda_L} \frac{1}{H' - z} 1_{\text{supp } V} \right\|, \tag{4.14}
\end{aligned}$$

where $\delta\Lambda_L := \Lambda_L \setminus \Lambda_{L-1}$ and the constant $C_{\text{GRE}} < \infty$ depends only on E_0 , the space dimension and the potentials V_0 and V . The operator norm in the last line of (4.14) is bounded by a Combes-Thomas estimate for operator kernels of resolvents, see e.g. [GK03, Thm. 1],

$$\left\| 1_\Gamma \frac{1}{H' - z} 1_{\Gamma'} \right\| \leq \frac{C_{\text{CT}}}{|y|} e^{-c_{\text{CT}} \text{dist}(\Gamma, \Gamma')|y|}. \tag{4.15}$$

It holds for all cubes $\Gamma, \Gamma' \subset \mathbb{R}^d$ of side length 1 and all z in some bounded subset of \mathbb{C} , which we choose as $\text{supp}(h) \times [-3, 3]$. The constants $C_{\text{CT}}, c_{\text{CT}} \in]0, \infty[$ in (4.15) can be chosen to depend only on E_0 , the space dimension and the potentials V_0 and V . Now, we assume $n \geq 2$, cover $\text{supp}(V)$ and $\delta\Lambda_L$ by unit cubes and apply the bounds (4.15) and (4.10) to (4.14). In this way we infer the existence of a constant $\tilde{C}_> \in]0, \infty[$, which is independent of L and t , such that

$$\begin{aligned}
\|D_L^>(t)\| &\leq \tilde{C}_>(E_0 + 1) Q_n(t/L^b) L^{d-1+b(n+1)} \int_{L^{-1+\varepsilon}}^3 dy y^{n-2} e^{-c_{\text{CT}} Ly/2} \\
&\leq C_> Q_n(t/L^b) L^{d-1+b(n+1)} e^{-c_{\text{CT}} L^\varepsilon/2} \tag{4.16}
\end{aligned}$$

for all $t \geq 0$ and all $L > L_n$, where L_n depends only on $\text{supp}(V)$ and $C_> := 3^{n-1}(n-1)^{-1}\tilde{C}_>(E_0 + 1)$.

Combining (4.5), (4.6), (4.11) – (4.13) and (4.16), we obtain the estimate

$$\int_0^{L^a} dt t K_L^{(2)}(t) \leq \left(C_{<} L^{b+n(-1+\varepsilon+b)} + C_{>} L^{d-1+b(n+1)} e^{-c_{\text{CT}} L^\varepsilon/2} \right) \Phi_n(L), \quad (4.17)$$

for all $L > L_n$, where

$$\begin{aligned} \Phi_n(L) &:= \int_0^{L^a} dt t Q_n(t/L^b) \operatorname{tr} \left(\sqrt{V} g_L^t(H_L) \sqrt{V} \right) \\ &\leq L^a Q_n(L^{a-b}) \int_0^\infty dt \operatorname{tr} \left(\sqrt{V} g_L^t(H_L) \sqrt{V} \right) \end{aligned} \quad (4.18)$$

The map $\nu_L : B \mapsto \operatorname{tr} (\sqrt{V} 1_B(H_L) \sqrt{V})$, where $B \in \text{Borel}(\mathbb{R})$, defines a Borel measure on \mathbb{R} . In this context we note that

$$\hat{\nu}(B) := \sup_{L>1} \nu_L(B) < \infty \quad \text{for } B \in \text{Borel}(\mathbb{R}) \text{ with } \sup B < \infty, \quad (4.19)$$

as can be seen from bounding the spectral projection in terms of the semi-group, $1_B(x) \leq \exp\{-(x - \sup B)\}$, $x \in \mathbb{R}$, and an explicit estimate using the Feynman-Kac representation, see e.g. [BHL00, Thm. 6.1]. From (4.19) and (3.33) we get the following estimate for the integral in the second line of (4.18)

$$\begin{aligned} &\int_0^\infty dt \operatorname{tr} \left(\sqrt{V} g_L^t(H_L) \sqrt{V} \right) \\ &= \int_{[-E_0-1, -L^{-b}]} d\nu_L(x) \frac{\chi_L^-(-x)}{(-x)} + \int_{[-L^{-b}, 0]} d\nu_L(x) \frac{\chi_L^-(-x)}{(-x)} \\ &\leq L^b \nu_L([-E_0-1, -L^{-b}]) + c_0 L^b \nu_L([-L^{-b}, 0]) \\ &\leq (1 + c_0) \hat{\nu}([-E_0-1, 0]) L^b =: \Xi L^b. \end{aligned} \quad (4.20)$$

Taken together, (4.17), (4.18) and (4.20) imply

$$\begin{aligned} \int_0^{L^a} dt t K_L^{(2)}(t) &\leq \Xi Q_n(L^{a-b}) L^{a+b} \left(C_{<} L^{b+n(-1+\varepsilon+b)} \right. \\ &\quad \left. + C_{>} L^{d-1+b(n+1)} e^{-c_{\text{CT}} L^\varepsilon/2} \right) \end{aligned} \quad (4.21)$$

for every $L > L_n$. We recall that $0 < \varepsilon < 1 - b$. Therefore we can choose n large enough as to ensure

$$a + 2b + n(-1 + b + \varepsilon) < 0. \quad (4.22)$$

Since we assumed $a < b$, we conclude that

$$\int_0^{L^a} dt t K_L^{(2)}(t) = o(1) \quad (4.23)$$

as $L \rightarrow \infty$. The same holds true for $K_L^{(1)}$ by an analogous argument. Thus, we have shown (4.2). \square

5. Proof of Lemma 3.15

We rewrite the difference in the integrand in (3.37) as $\chi_L^-(x)[\chi_L^+(y) - 1_{[E, E_0]}(y)] + 1_{[E, E_0]}(y)[\chi_L^-(x) - 1_{[-E_0, E]}(x)]$. Extending the t -integral in (3.37) up to $+\infty$, the two error terms from removing the smoothing where $x \in [-E_0 - 1, -E_0]$ or $y \in [E_0, E_0 + 1]$ are seen to be bounded from above by

$$\int_{\mathbb{R}^2} \frac{d\mu_{ac}(x, y)}{(y - x)^2} \left[1_{[-E_0-1, E]}(x) 1_{[E_0, E_0+1]}(y) + 1_{[-E_0-1, -E_0]}(x) 1_{[E, E_0+1]}(y) \right]. \quad (5.1)$$

This L -independent expression is finite, because $y - x \geq 1$ in the compact support of the integrand thanks to $E \in [-E_0 + 1, E_0 - 1]$. Thus, this error is of order $O(1)$ as $L \rightarrow \infty$.

In order to estimate the two error terms in (3.37) from removing the smoothing where $x \in [E - L^{-b}, E]$ or $y \in [E, E + L^{-b}]$, we use the inequality (3.12) for the Lebesgue density of μ_{ac} together with the elementary estimate $te^{-t\zeta} \leq \zeta^{-1}$ for $t, \zeta > 0$. This gives the upper bound

$$L^a \int_{\mathbb{R}^2} dx dy \frac{\gamma_1(x) \gamma_2(y)}{y - x} \left[1_{[-E_0-1, E]}(x) 1_{[E, E+L^{-b}]}(y) + 1_{[E-L^{-b}, E]}(x) 1_{[E, E_0]}(y) \right]. \quad (5.2)$$

But this bound is of order $O(L^{a-b} \ln L)$, as follows from the next lemma because E is a Lebesgue point of both γ_1 and γ_2 . Since $b > a$, the proof is complete. \square

Lemma 5.1. *Let $A > 0$ and $\kappa_1, \kappa_2 \in L^1_{loc}(\mathbb{R})$. Assume that 0 is a Lebesgue point of both κ_1 and κ_2 . Then we have*

$$\int_{-A}^0 dx \int_0^\eta dy \frac{\kappa_1(x) \kappa_2(y)}{y - x} = O(\eta \ln \eta) \quad (5.3)$$

as $\eta \downarrow 0$.

PROOF. W.l.o.g. we assume $\kappa_1, \kappa_2 \geq 0$. Let $\eta \in]0, 1[$. We start with an estimate for the y -integral for given $x < 0$. The function $[0, \eta] \ni y \mapsto \int_0^y d\zeta \kappa_2(\zeta)$ is absolutely continuous by the fundamental theorem of calculus for Lebesgue integrals. Therefore we can apply integration by parts to conclude

$$\begin{aligned} \int_0^\eta dy \frac{\kappa_2(y)}{y - x} &= \frac{\int_0^y d\zeta \kappa_2(\zeta)}{y - x} \Big|_{y=0}^{y=\eta} + \int_0^\eta dy \frac{\int_0^y d\zeta \kappa_2(\zeta)}{(y - x)^2} \\ &\leq C_2 \left(\frac{\eta}{\eta - x} + \int_0^\eta dy \frac{y}{(y - x)^2} \right) = C_2 \ln(1 - \eta/x), \end{aligned} \quad (5.4)$$

where $C_2 := \sup_{y \in]0, 1[} y^{-1} \int_0^y d\zeta \kappa_2(\zeta) \in]0, \infty[$ is finite, because 0 is a Lebesgue point of κ_2 .

The function $[-A, 0] \ni x \mapsto -\int_x^0 d\zeta \kappa_1(\zeta)$ is absolutely continuous so that we can use partial integration to compute the x -integral of κ_1 times the

bound (5.4)

$$\begin{aligned} & \int_{-A}^{-\varepsilon} dx \kappa_1(x) \ln(1 - \eta/x) \\ &= \left(- \int_x^0 d\zeta \kappa_1(\zeta) \right) \ln(1 - \eta/x) \Big|_{x=-A}^{x=-\varepsilon} + \int_{-A}^{-\varepsilon} dx \frac{\eta/x^2}{1 - \eta/x} \int_x^0 d\zeta \kappa_1(\zeta), \end{aligned} \quad (5.5)$$

where we introduced $\varepsilon \in]0, A[$ to exclude the singularity of the logarithm (and have dropped the constant C_2). Since 0 is a Lebesgue point of κ_1 , we infer the existence of a finite constant $C_1 := \sup_{x \in]-A, 0[} (-x)^{-1} \int_x^0 d\zeta \kappa_1(\zeta) \in]0, \infty[$ and can perform the limit $\varepsilon \downarrow 0$ to get

$$\begin{aligned} \int_{-A}^0 dx \int_0^\eta dy \frac{\kappa_1(x) \kappa_2(y)}{y - x} &\leq C_1 C_2 \left[A \ln(1 + \eta/A) + \int_{-A}^0 dx \frac{\eta}{\eta - x} \right] \\ &= C_1 C_2 [A \ln(1 + \eta/A) + \eta \ln(1 + A/\eta)]. \end{aligned} \quad (5.6)$$

This implies (5.3). \square

Appendix. Relation to Scattering Theory

In this section we conduct the proofs of Theorem 2.4 and Corollary 2.6. The goal is to express the decay exponent $\gamma(E)$ of the overlap in Theorem 2.2 in terms of quantities from Scattering Theory.

PROOF OF THEOREM 2.4. We recall that $V_0 = 0$ in this theorem. Hence, (2.1) implies that both H and H' have purely absolutely continuous spectrum with $\sigma(H) = \sigma(H') = [0, \infty[$. Moreover, we know from (3.9) and Lemma 3.2 that

$$\gamma(E) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \operatorname{tr}(\sqrt{V} P_E^\varepsilon V \Pi_E^\varepsilon \sqrt{V}) \quad (A.1)$$

exists for Lebesgue-a.e. $E \in \mathbb{R}$. This implies in particular that $\gamma(E) = 0$ for $E < 0$.

The integral kernels of the spectral projections P_E^ε and Π_E^ε can be represented as

$$\begin{aligned} P_E^\varepsilon(x, y) &= \frac{1}{2(2\pi)^d} \int_0^\infty d\lambda \lambda^{\frac{d-2}{2}} 1_{I_{E,\varepsilon}}(\lambda) \int_{\mathbb{S}^{d-1}} d\Omega(\omega) e^{i\sqrt{\lambda}y \cdot \omega} e^{-i\sqrt{\lambda}x \cdot \omega}, \\ \Pi_E^\varepsilon(x, y) &= \frac{1}{2(2\pi)^d} \int_0^\infty d\lambda \lambda^{\frac{d-2}{2}} 1_{I_{E,\varepsilon}}(\lambda) \int_{\mathbb{S}^{d-1}} d\Omega(\omega) \overline{\psi(y; \omega, \lambda)} \psi(x; \omega, \lambda) \end{aligned} \quad (A.2)$$

for all $x, y \in \mathbb{R}^d$, where $I_{E,\varepsilon} :=]E - \varepsilon/2, E + \varepsilon/2[$, $x \cdot \omega$ stands for the Euclidean scalar product of x with ω (viewed as a unit vector in \mathbb{R}^d) and $\psi : \mathbb{R}^d \times \mathbb{S}^{d-1} \times [0, \infty[\rightarrow \mathbb{C}$ are the generalized eigenfunctions of $H' = -\Delta + V$ due to Ikebe and Povzner, see Sect. 1.3 in [Yaf00] and references therein.

In particular, $\psi(\cdot; \omega, \lambda)$ is a solution to the Lippmann-Schwinger equation. Inserting (A.2) in (A.1) gives

$$\begin{aligned} \gamma(E) &= \frac{E^{d-2}}{4(2\pi)^{2d}} \int_{\mathbb{S}^{d-1}} d\Omega(\omega) \int_{\mathbb{S}^{d-1}} d\Omega(\theta) \left| \int_{\mathbb{R}^d} dx V(x) e^{i\sqrt{E}x \cdot \theta} \psi(x; \omega, E) \right|^2 \\ &= \frac{E^{(d-1)/2}}{(2\pi)^{d+1}} \int_{\mathbb{S}^{d-1}} d\Omega(\omega) \int_{\mathbb{S}^{d-1}} d\Omega(\theta) |a(\theta, \omega; E)|^2 \end{aligned} \quad (\text{A.3})$$

for Lebesgue-a.a. $E \geq 0$, where the scattering amplitude is defined by

$$a(\theta, \omega; E) := -e^{i\pi(d-3)/4} \frac{E^{(d-3)/4}}{2(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^d} dx e^{-i\sqrt{E}x \cdot \theta} V(x) \psi(x; \omega, E), \quad (\text{A.4})$$

see [Yaf00, Eq. (1.17)]. Following [Yaf00, Sect. 8.5], the right-hand side of (A.3) can be rewritten in terms of the total scattering cross-section $\sigma(E, \omega) = \int_{\mathbb{S}^{d-1}} d\Omega(\theta) |a(\theta, \omega; E)|^2$ or in terms of the scattering matrix $S(E)$, which proves the lemma. \square

PROOF OF COROLLARY 2.6. The corollary is concerned with the special case $d = 3$ and V spherically symmetric. In particular, the total scattering cross-section $\sigma(E) = \sigma(E, \omega)$ does not depend on the incident direction $\omega \in \mathbb{S}^2$ for any $E \geq 0$. Following [RS79, Eq. (111)], we can rewrite $\sigma(E)$ in terms of the partial wave amplitudes $f_\ell(E)$ so that (2.12) becomes

$$\gamma(E) = \frac{E}{(2\pi)^4} \int_{\mathbb{S}^2} d\Omega(\omega) \sigma(E) = \frac{E}{\pi^2} \sum_{\ell=0}^{\infty} (2\ell+1) |f_\ell(E)|^2 \quad (\text{A.5})$$

for Lebesgue-a.a. $E \geq 0$. Now, [RS79, Eq. (112b)] finishes the proof. \square

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